*On Independent Definitions of the Functions $\log(x)$ and e^x.**

BY EMORY McCLINTOCK.

Twelve years ago there appeared in the second volume of the *American Journal of Mathematics* "An Essay on the Calculus of Enlargement," in which I presented, and urged the acceptance of, a certain unified view of several branches of mathematical science. Giving the name of Enlargement to that operation (E^h) by which $\phi(x)$ becomes $\phi(x+h)$, corresponding to the symbolic equation $E^h\phi(x) = \phi(x+h)$, I remarked that the symbolic algebra of the rational functions of E, commonly known as the Calculus of Finite Differences, and the symbolic algebra of the logarithmic functions of E, commonly known as the Differential Calculus, were really parts of one symbolic algebra of the functions of E, for which I suggested the name of Calculus of Enlargement. After observing that the connecting link between the theory of differentiation and the theory of finite differences had long been thought to be furnished by the equation $E = e^D$ where D means $\frac{d}{dx}$, the symbol of differentiation, I argued that the converse view, $D = \log(E)$, would be preferable because "of the two operations, the simpler should be defined the earlier." Plainly, $E^h\phi(x) = \phi(x+h)$ is a simpler statement than $D\phi(x) = \lim[\phi(x+h) - \phi(x)]/h$ when h is indefinitely reduced. "These operations, E and D, are functions of each other, and whichever is defined last must be expressed in terms of the other." "The Calculus of Enlargement regards E as the fundamental symbol, and takes cognizance of other symbols only in case they are, and because they are, functions of E." "The algebra of the functions of E is subject to all the laws of ordinary algebra; and the theory of differentiation is that part of the calculus which corresponds to the theory of logarithms in algebra."

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* An abstract of this paper was read before the New York Mathematical Society on March 6, 1891.

That the acknowledged correspondence between symbolic algebra and ordinary algebra might be brought out in the strongest light, it was then urged that the customary mode of presenting the theory of logarithms be so changed as to make logarithms, as such, more intelligible. The prevailing obscurity was illustrated by quoting De Morgan's sweeping statement that "the only definition of $\log(x)$ used in analysis is y, where $e^y = x$." At first sight this definition is not satisfactory. It is true that, by convention, e^y means a certain limit, or a certain series, and not, except when y is a real quantity, a power of the constant e. While acknowledging the correctness of such customary indirect definitions of $\log(x)$, I ventured to propose concurrently other possible definitions, and among them the known equation $\log(x) = \lim (x^h - 1)/h$, which has since, I am glad to see, been mentioned as a feasible definition by Mr. Glaisher in the article "Logarithms" in the Encyclopædia Britannica. The definitions which were then suggested concurrently for $\log(x)$ all tended to throw light on the nature of the logarithm, and were all, of course, susceptible of subsequent identification. The discussion of logarithms as such was not, however, essential to the chief object then in hand, and for that reason, perhaps, I failed to carry out at that time the notion of concurrent definition to its logical consequences.

In truth, what we may call "the method of concurrent definition" has probably not hitherto been formulated as a scientific method of procedure. The long-standing idea of a definition is that it indicates the thing defined, and that from it, as an unchangeable basis, all other properties must be deduced. We are at liberty to begin with any given relation of a function as a definition, but having once chosen it, we are to adhere to it, since the very essence of a definition appears to be that it at least is definite and unchangeable, so that two simultaneous definitions of one idea would seem logically monstrous. To put the matter in form, then, let us say that "the method of concurrent definition" comprises the definition of $f_1(x)$, with an illustration of its nature, and the separate and independent definition of $f_2(x)$ in like manner, followed by proof that $f_1(x)$ and $f_2(x)$ are identical. Intrinsically, no process can be more logical; and when we come to reflect upon it, we shall find that we have, upon occasion, been practising it all our lives.

Take for instance the best known of all functions, the binomial function $f_1(x, y, m) = (x + y)^m$. Starting from this definition, Newton and his followers for a century undertook to deduce from it the binomial series, with more or less

success, until Euler brought out a better statement of the case, employing in substance the principle of concurrent definition. Denoting the series, say, as $f_2(x, y, m)$, he took it up as an independent function, proved certain properties, and then identified f_2 with f_1.

In another paper (*ante*, p.), I have supplied an algebraic proof of a series equivalent to $\log(x)$, for which series, when first presented as an expansion, I could give no better demonstration than that afforded by Lagrange's theorem. The new proof begins with the consideration of the series as a separate function and ends with its identification with known equivalents of $\log(x)$.

It used to be customary to expand e^x, or $\lim(1+h)^{x/h}$ when h tends towards 0, by means of the binomial series in order to obtain the exponential series $1+x+x^2/2!+\ldots.$ A better way was devised by Cauchy, beginning with the latter series as a separate subject for examination, deducing its properties, and finally identifying it with e^x. The series is now, indeed, adopted by some of the highest authorities as the original definition of the symbol e^x, sometimes written $\exp(x)$; although others adhere to the limit as the proper definition. Thus in the article "Function" in the Encyclopædia Britannica the series is employed as the definition by Cayley, while in the article "Trigonometry" the limit is employed by Hobson. There is no occasion for controversy. The limit is a definable function possessed of certain properties which may be discussed; the series is another, and the two may be readily identified. Yet I have not met with any distinct announcement of the utility of assigning coordinate rank to these two methods of presenting e^x; those who employ the series having apparently no interest in the limit, and those who begin with the limit appearing to regard the series as a subsidiary deduced expression, for the acquisition of which Cauchy's method is, as he meant it to be, merely an incidental device.

To illustrate more clearly the two views now prevalent, the following summaries of the definitions contained in the Encyclopædia articles just mentioned will be found interesting. The article "Trigonometry" defines e^x as $\lim(1+x/m)^m$, where m tends towards infinity, and $e^{x+\iota y}$ as $\lim(1+[x+\iota y]/m)^m$, so that, putting

$$1+x/m=r\cos\theta, \text{ and } y/m=r\sin\theta,\ e^{x+\iota y}=\lim r^m(\cos m\theta+\iota\sin m\theta),$$

by De Moivre's theorem. Since

$$r^2=r^2\cos^2\theta+r^2\sin^2\theta,\ \lim r^m=\lim(1+2x/m+x^2/m^2+y^2/m^2)^{m/2}$$
$$=\lim(1+2x/m)^{m/2}=e^x.$$

Also, $\lim m\theta = \lim m \operatorname{arc}\tan y/(x+m) = \lim my/(x+m) = y$.

Hence $e^{x+\iota y} = e^x(\cos y + \iota \sin y)$. The expansion of e^x is assumed known, by algebra. On the other hand, the article "Function" defines $\exp(x)$ to be $1 + x + x^2/2! + \ldots.$, where x need not be real, $\cos(x)$ to be $1 - x^2/2! + \ldots.$, and $\sin(x)$ to be $x - x^3/3! + \ldots.$, and deduces the theory of exponential, circular, hyperbolic, and logarithmic functions from these definitions without mention of the limit-expression.

If we attempt to embrace both views at once by declaring both definitions useful, we shall have on the one hand $f_1(x) = \lim(1+hx)^{1/h}$, as h tends towards 0, and on the other hand $f_2(x) = 1 + x + x^2/2! + \ldots.$ It is easy to prove that $[f_1(1)]^x = f_1(x)$, and that $[f_2(1)]^x = f_2(x)$, so that $f_1(x)$ and $f_2(x)$ are identical if $f_1(1) = f_2(1)$. Let $f_1(1) = f_2(c)$. Let each of these be raised to the power h, let 1 be subtracted, and let the respective remainders be divided by h, after which let h be reduced indefinitely in value; the result in the one case is 1, in the other c, for $[f_2(c)]^h = f_2(ch)$, so that $c = 1$, and the functions are identical. We shall find thus that the connecting link between these two definitions of e^x is y, where $x = \lim(y^h - 1)/h$. Hitherto, I think, the two expressions have been connected, whenever necessary, by employment of the binomial series.

Of the six equations following, the third, fourth, and fifth represent known definitions, and the others represent some of the known identities which, in the earlier paper, I suggested for use in the future as definitions:

$$\phi_1(x) = \log(x) = \lim(x^h - 1)/h, \tag{1}$$
$$\phi_2(x) = \log(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \ldots., \tag{2}$$
$$\log(x) = y, \text{ where } e^y = x, \tag{3}$$
$$f_1(y) = \exp(y) = e^y = \lim(1+yh)^{1/h} = \lim(1+k)^{y/k}, \tag{4}$$
$$f_2(y) = \exp(y) = e^y = 1 + y + y^2/2! + \ldots., \tag{5}$$
$$\exp(y) = e^y = x, \text{ where } y = \log x. \tag{6}$$

It is known that the exponential property, $e^u e^v = e^{u+v}$, is a direct result of the definition (4), and that it may be derived, as by Cauchy, from (5), by multiplication of the two series. I shall show similarly that the logarithmic property, $\log(x^n) = n\log(x)$, is a direct result of (1), and that it may also be derived from (2). I have already noted that (1) is the natural connecting link between (4) and (5); and I shall show similarly that (4) is the natural connecting link between (1) and (2).

There is indeed no good reason for the priority hitherto given to e^y in relation to $\log(x)$. It is probably still true that "the only definition of $\log(x)$ used in analysis is y, where $e^y = x$," but we are at liberty to change the custom if it is not a good one. The first notion of logarithms was due to the observed fact that powers form a geometrical series while the numbers denoting the powers form an arithmetical series; and the first practical notion that any child obtains is that the "common logarithm" is the index of that power of 10 which is equal to the "number corresponding." Our early familiarity with the equation $10^{\lambda(n)} = n$, where λ represents the "common logarithm," is a sufficient explanation of the historical fact that, by custom, $\log(x)$ means "y, where $e^y = x$." The origin of the custom is natural and obvious, but the question is whether it is a necessary custom. What is e, and what is y? The analytic logarithm defined as "y, where $e^y = x$," is an incommensurable power, whatever that may be—for all the powers at first known to us are commensurable—of an incommensurable quantity. Is this really the simplest explanation we can get of $\log(x)$, even when x is real and positive? But the use of the functional symbols e^y, $\log(x)$, has gone far beyond that restriction. We have seen how e^y has to be defined when y is complex; the idea of powers, at first so natural, so apparently simple, is flatly abandoned. We shall now see that for all quantities, real or complex, $\log(x)$ is a function at least as easily apprehended as e^y; while it is known—and this, as the reader will have understood, is the strongest motive with the present writer—that in symbolic algebra E is simpler than D, and therefore $D = \log(E)$ should have priority, as an analytical statement, over $E = e^D$. For the presentation of the true definition of differentiation, $D = \log(E)$, we need to have some better algebraic definition of the logarithm than "y, where $e^y = x$." There is a recognized mechanical incongruity when the cart precedes the horse.

Referring to (1), let us adopt as a definition

$$\log(x) = \lim (x^h - 1)/h, \tag{1}$$

where h tends towards 0. Let $x_1 = x^h - 1$, and $u_1 = u^h - 1$, so that $\lim x_1 = 0$. Then

$$\begin{aligned} \log(xu) &= \lim (x^h u^h - 1)/h = \lim \big([1 + x_1][1 + u_1] - 1\big)/h \\ &= \lim x_1/h + \lim u_1/h + \lim x_1 . \lim u_1/h \\ &= \log(x) + \log(u). \end{aligned} \tag{7}$$

This fundamental property of logarithms is thus shown to be obtained at once from (1) as a definition. It will be proved (Appendix A) that from (7) we have readily

$$\log (x^n) - n \log (x), \tag{8}$$

where n is any commensurable number, and where x is any symbol subject to algebraic laws.

If e is the number whose logarithm is 1, we find from (8) that

$$\log (e^n) = n. \tag{9}$$

This is equivalent to (6) as an inverse definition of e^n. But, more directly from (1), if $y = \log x$, we have $y = \lim x_1/h$, and since $x^h = 1 + x_1$,

$$x = (1 + x_1)^{1/h} = \lim (1 + hy)^{1/h} = \lim (1 + k)^{y/k}, \tag{10}$$

if $k = hy$; and either of these expressions may be denoted, concurrently, by the symbol e^y or $\exp(y)$. This is in fact Schlömilch's well known method of introducing the function e^y;* and that one of the most satisfactory current explanations of e^y as a limit should include the prior introduction of the function $(x^h - 1)/h$ is, I think, a circumstance which gives strong support to the analytic order now advocated.

We shall now see that the use of (1) as the definition of $\log (x)$ is particularly effective when x is complex. Let $x = r(\cos\theta + \iota \sin\theta)$, taking θ between $+\pi$ and $-\pi$. Observing that $\lim r^h = 1$, that $\lim \cos h\theta = 1$, and that $\lim \sin h\theta/h\theta = 1$, and remembering that $(\cos\theta + \iota\sin\theta)^h = \cos h\theta + \iota \sin h\theta$, we have

$$\begin{aligned} \log (x) &= \lim [r^h (\cos h\theta + \iota \sin h\theta) - 1]/h \\ &= \lim (r^h - 1)/h + \lim \iota\theta \sin h\theta/h\theta \\ &= \log (r) + \iota\theta, \end{aligned} \tag{10}$$

a result hitherto obtained by inverse processes. If we permit the value of θ to go beyond the limits assigned, we shall have multiple values for $\log (x)$, corresponding to the usual statement of such values.

Nor must it be supposed that real values of $\log (x)$ are less intelligible when (1) is employed as a definition than when, as usual, $\log (x)$ is presented as "y, where $e^y = x$." We are familiar with e, regarded as the limit of $(1 + k)^{1/k}$ when

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*Zeitschrift für Mathematik, III, 387; Algebraische Analyse, 5th ed., 35–39; see also Chrystal, Algebra, II, 79. Schlömilch employs the function $(x^h - 1)/h$, but not its known limit, $\log (x)$, in presenting e^y. On the contrary, he adheres to the current view of $\log (x)$ as "y, where $e^y = x$."

k tends towards 0; we know how that function increases, and how $(1-k)^{-1/k}$ diminishes, as k diminishes, both tending towards the same numerical limit between 2 and 3. We shall now see that $\lim (x^h - 1)/h$ is at least as simple an idea, at least as easy to comprehend and illustrate, as $\lim (1+k)^{y/k}$, its inverse.

Let us consider the functions $(x^h - 1)/h$ and $(1 - x^{-h})/h$, which for distinction let us call the upper fraction and the lower fraction respectively. Here x and h are positive, and h is commensurable and not greater than 1. These fractions are definite continuous functions of x, devoid of mystery, positive when $x > 1$, negative where $x < 1$, and becoming 0 when $x = 1$. When $x = \infty$, $x^h = \infty$, $x^{-h} = 0$; so that as x varies from 0 to ∞, the upper fraction varies from $-h^{-1}$ to ∞, and the lower fraction from $-\infty$ to h^{-1}, each passing through 0 when $x = 1$. The upper fraction is equal to the lower fraction multiplied by x^h, so that the smaller the value of h the less their difference. If, for example, $x = 4096$, and if certain values be assigned to h, we find corresponding values for the fractions as follows:

	$h=1.$	$h=\frac{1}{2}.$	$h=\frac{1}{3}.$	$h=\frac{1}{4}.$	$h=\frac{1}{6}.$	$h=\frac{1}{12}.$
$\dfrac{x^h-1}{h}$	4095	126	45	28	18	12,
$\dfrac{1-x^{-h}}{h}$	$\dfrac{4095}{4096}$	$\dfrac{63}{32}$	$\dfrac{45}{16}$	$\dfrac{7}{2}$	$\dfrac{9}{2}$	6.

A well known algebraic inequality,

$$(x^h - 1)/h > (x^{h_1} - 1)/h_1, \tag{11}$$

where $h_1 < h$ teaches us that as h diminishes the upper fraction diminishes in value. The following inequality, presumably new, is readily derivable from (11):

$$(1 - x^{-h})/h < (1 - x^{-h_1})/h_1. \tag{12}$$

This shows that as h diminishes the lower fraction increases in value. But neither of them, one diminishing and the other increasing, can pass the other, for they have the same limit, because their ratio is x^h, of which the limit is 1. This common limit is denoted by the symbol $\log(x)$. A simple, and probably novel, proof of (11) and (12) will be given later (Appendix B).

But we need not rest here. I showed in the earlier paper that since $\lim u^h = 1$, u being any function of x, we may modify (1) thus:

$$\log(x) = \lim u^h (x^h - 1)/h. \tag{13}$$

In particular, if $u = x^{-ah}$,

$$\log(x) = \lim (x^{h-ah} - x^{-ah})/h.^* \tag{14}$$

The upper fraction is, in the limit, that special case of (14) in which $a = 0$, and the lower fraction is that special case in which $a = 1$. A third important case, which I indicated at the same time, is that in which $a = \frac{1}{2}$, the limit of a remarkable function which we may call the central fraction:

$$\log(x) = \lim (x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h. \tag{15}$$

This central fraction is the geometric mean between the upper and lower fractions, and may be illustrated in connection with them by employing the same example as before, $x = 4096$:

	$h=1.$	$h=\frac{1}{2}.$	$h=\frac{1}{3}.$	$h=\frac{1}{4}.$	$h=\frac{1}{6}.$	$h=\frac{1}{12}.$
$\dfrac{x^h - 1}{h}$	4095	126	45	28	18	12,
$\dfrac{x^{\frac{1}{2}h} - x^{-\frac{1}{2}h}}{h}$	$\dfrac{4095}{64}$	$\dfrac{63}{4}$	$\dfrac{45}{4}$	$7\sqrt{2}$	9	$6\sqrt{2}$,
$\dfrac{1 - x^{-\frac{1}{2}h}}{h}$	$\dfrac{4095}{4096}$	$\dfrac{63}{32}$	$\dfrac{45}{16}$	$\dfrac{7}{2}$	$\dfrac{9}{2}$	6.

The limit to which all three fractions converge is $8.317 = \log 4096$, and, as might be expected, the central fraction gives, for any small value of h, by far the best approximation of the three. The fact that

$$(x^h - 1)/h > (x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h > (1 - x^{-h})/h \tag{16}$$

may be shown at once from the existence of the factor $x^{\frac{1}{2}h}$, since x and h are both positive, and all three fractions are positive or negative simultaneously. I have found, however, and shall prove (Appendix C), that, in all cases wherein $x > 1$,

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h > (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1, \tag{17}$$

h_1 being smaller than h; and also that, when $0 < x < 1$,

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h < (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1. \tag{18}$$

Taking (17) with (12), therefore, we perceive that, when $x > 1$, the central fraction diminishes in value with h, while the lower fraction increases, so that their common limit $\log(x)$ lies between any value of the central fraction and any

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* See also "Algebraic Proof of a Certain Series," *ante* p.

value of the lower fraction. On the other hand, when $0 < x < 1$, taking (18) with (11), we perceive that the central fraction increases algebraically (diminishing numerically because negative) when h diminishes, while the upper fraction diminishes algebraically (increasing numerically), so that their common limit $\log(x)$ lies between any value of the central fraction and any value of the upper fraction.

We are thus enabled, by means of the definition $\log(x) = \lim(x^h - 1)/h$, to obtain a clear idea of the real nature of a logarithm (impossible to obtain from the customary "y, where $x = e^y$"), as well as to deduce the usual properties of logarithms with the utmost ease. Employing this definition concurrently with either of the usual definitions of $\exp(y)$, whether defined as a limit or as a series, we may thus obtain a full and comprehensive view of the whole subject of logarithms and exponentials; and indeed, as we have seen, the readiest method of connecting the limit and the series expressing $\exp(y)$ is to resort to the limit which expresses $\log(x)$.

Perhaps the reader is still naturally reluctant to admit that any mode of presenting logarithms can take the place of that afforded by the analogy of common logarithms, the meaning of which is represented by the equation $10^{\lambda(n)} = n$. But the principle of concurrent definitions permits us to retain all rays of light derived from every quarter. The equation $\log(x) = y$, where $e^y = x$, is a truth which we may always use to the extent of its worth. Yet common logarithms were not originally discovered or computed by means of what seems to us the simple equation $10^{\lambda(n)} = n$. If we multiply both sides of (7) by k, a constant, we have, if $\phi(x) = k\log(x)$, $\phi(xu) = \phi(x) + \phi(u)$. That is to say, the logarithmic property which facilitates multiplication by prepared tables is possessed by any function $k\log(x)$ as well as by $\log(x)$. In Napier's tables, $k = -10{,}000{,}000$, and his "artificial numbers" (such was the circuitous route by which he approached the subject) contained another constant, say c, so that each of them might be represented by the formula $k\log(z) + c$. They could therefore be used only under restrictions. Subsequently, Briggs and Napier between them devised, and Briggs carried out, the idea of computing $p = k\log(z)$, where $k = 1/\log(10)$, the relation $z = 10^p$ affording unique advantages in practice. That this relation exists—or, more generally, $z = x^p$, where we define p as $k\log(z)$ and k as $1/\log(x)$—follows at once from (8); for $\log(x^p) = p\log(x)$ $= \log(z)\log(x)/\log(x) = \log(z)$; that is, $x^p = z$. It is a matter of historic

interest that Briggs's computation of the "common logarithm" of 2 was effected precisely upon the lines just indicated. He calculated $\log(2)$ and $\log(10)$ separately by the formula $\log(x) = \lim (x^h - 1)/h$, taking h very small by many extractions of square roots, and then obtained the "common logarithm" of 2 by multiplying $\log(2)$ by the modulus $1/\log(10)$.*

Reverting to our first list of possible definitions of $\log(x)$ and e^y, numbered (1) to (6), we find (2) still remaining to be discussed. The well-known series

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots = \psi(z), \tag{19}$$

may be used, as suggested in (2), taking $z = x - 1$ as the definition of $\log(x)$ whenever the series is real and summable, say when $-1 < z \overline{\gtrless} 1$. If, in all other real cases, we define $\log(x)$ to be $-\log(x^{-1})$, I shall prove (Appendix D) that the series for $\log(x)$ added to the series for $\log(u)$ forms a sum equivalent to the series for $\log(xu)$, from which, as before, the essential properties of logarithms will follow. This is, in effect, accomplishing for the logarithmic series what is accomplished for the exponential series when the series for e^y and that for e^w are multiplied together to form a product equivalent to the series for e^{y+w}, according to Cauchy's well-known method. The same proof holds good for all cases in which x is not a real quantity (say either a complex quantity or a symbol of operation), and in which the several series involved are respectively interpretable.

Having once established that $\log(x) = \lim (x^h - 1)/h$, we shall find, on the one hand, that this definition of $\log(x)$ produces immediately, and with the greatest ease, the logarithmic series, if we substitute the binomial series $(1+z)^h$ for x^h, so that the binomial series forms a connecting link between the limit-definition and the series-definition of $\log(x)$. On the other hand, a different connecting link may be found in the limit customarily employed to define e^x. Assuming that we know that $\phi_2(x) + \phi_2(u) = \phi_2(xu)$, where, as in (2), $\phi_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \ldots$, and therefore that $\phi_2(x^n) = n\phi_2(x)$, it follows that $\phi_2(x) = \lim (x^h - 1)/h$, called $\log(x)$. For, let us suppose $\phi_2(x) = \log(x) . f(x)$, where $f(x)$ is some unknown function. Then $\phi_2(x^n) = \log(x^n) . f(x^n) = n\phi_2(x) = n\log(x) . f(x)$, and we know by (8) that $\log(x^n) = n\log(x)$. Dividing one of these by the other, we have $f(x^n) = f(x)$, so that $f(x)$ is independent of the

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* Encyc. Brit.: "Logarithms."

value of x, since x^n may have any value; hence $f(x)$ is a constant, say c, so that $\phi_2(x) = c \log(x)$. Therefore

$$c \log(1 + hz) = hz - \tfrac{1}{2} h^2 z^2 + \ldots,$$

and $c \lim \log(1 + hz)/h = \lim(z - \tfrac{1}{2} hz^2 + \ldots) = z$. But $\lim \log(1 + hz)/h = \log e^z = z$, whence $c = 1$, and

$$\log(1 + z) = z - \tfrac{1}{2} z^2 + \tfrac{1}{3} z^3 - \ldots. \tag{20}$$

Although the definition of $\log(x)$ as a series is by no means so general in its nature or so satisfactory as the series-definition of e^y, the consideration of $\log(x)$ from this point of view will necessarily add to the broadness and clearness of our knowledge of the function. Indeed, for mere intelligibility, it may be remarked that such an expression, for example, as $\log(\tfrac{3}{2}) = \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 - \ldots = 0.405$ nearly, is really easier for the mind to apprehend than $\log(\tfrac{3}{2}) = \lim(\tfrac{3}{2}^h - 1)/h$, or than "$y$, where $e^y = \tfrac{3}{2}$."

It is not difficult to expand the logarithm of a complex quantity by means of the logarithmic series. If $x = u + \iota v$ and $u = \pm \mod u$, let $v/u = \pm z$, so that $x = (\pm 1 + \iota z).\mod u$, and $\log(x) = \log(\mod u) + \log(\pm 1 + \iota z)$. Here $\mod u$ is real and positive, and its logarithm may therefore be expressed by a series. As regards $\log(\pm 1 + \iota z)$, the case $\log(1 + \iota z)$, where $z^2 \overline{\gtrless} 1$, is well known, and the expansion is obtainable directly. If $z^2 > 1$, we may expand $\log(1 + \iota z)$ by taking y such that $(1 + \iota y)^2 = m(1 + \iota z)$, where m is real and positive, and here $y^2 < 1$, so that $\log(1 + \iota z) = 2 \log(1 + \iota y) - \log m$, which may be expressed in a series. Again, if the sign be negative, we may take u such that $(1 + \iota u)^2 = n(-1 + \iota z)$, where n is real and positive, and $\log(-1 + \iota z)$ may be expressed as $2 \log(1 + \iota u) - \log n$. The expression $\pm 1 + \iota z$ may be taken to represent a point in one of two perpendicular lines tangent to a circle whose radius is 1, in which case the unreal part of the logarithm represents the length of the arc cut off by a line from the point to the centre; and the unreal part of the series obtained is in the usual form of the series for an arc in terms of its tangent. That $\text{arc} \tan x = \tfrac{1}{2} \iota \log[(1 - \iota x)/(1 + \iota x)]$ is known.

Appendix A.—If $\phi(x)$ be a function such that $\phi(x) + \phi(u) = \phi(xu)$, then $\phi(x^n) = n\phi(x)$. For, $\phi(xuv) = \phi(xu) + \phi(v) = \phi(x) + \phi(u) + \phi(v)$, and similarly, if there are n such quantities, $\phi(xu \ldots w) = \phi(x) + \phi(u) + \ldots + \phi(w)$.

Taking $u, v, \ldots . w$ severally equal to x, it follows that, when n is any positive integer,

$$\phi(x^n) = n\phi(x). \tag{21}$$

Since $\phi(x) + \phi(1) = \phi(x)$, $\phi(1) = 0$, and $\phi(x) + \phi(x^{-1}) = \phi(1) = 0$; whence $\phi(x^{-1}) = -\phi(x)$, so that, writing x^{-1} for x in (21), we have $\phi(x^{-n}) = -n\phi(x)$, so that (21) is true when n is a negative integer. That it is also true when n is fractional, say when $n = p/q$, may be shown by writing $x^{1/q}$ for x and q for n, whence $\phi(x) = q\phi(x^{1/q})$; then, bearing this in mind, and again writing $x^{1/q}$ for x in (21), but p for n, we have $\phi(x^{p/q}) = p\phi(x^{1/q}) = p/q\,\phi(x)$. The symbol x is not restricted here to real values; it may be complex, or it may be a symbol of operation. On the other hand, n is real and commensurable.

Appendix B.—If x, q and r are real and positive quantities, the following inequality is always true:

$$qx^q(x^r - 1) > (x-1)\,qrx^q > r(x^q - 1). \tag{22}$$

To prove this when q and r are integers, we have only to recollect that $x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + \ldots . + x + 1)$; for $x^{r-1} + x^{r-2} + \ldots . + 1 >< r$, and $qx^q >< x^{q-1} + x^{q-2} + \ldots . + 1$, according as $x >< 1$; that is, according as the several members of (22) are all positive or all negative by reason of their common factor $x - 1$, the effect of a negative factor being to reverse the sign of inequality. Any such inequality proved true for integral exponents is necessarily true for fractional exponents, since we are at liberty to write $x^{1/p}$ for x. If we add $qx^q - q$ to both sides of (22), we have $q(x^{q+r} - 1) > (q+r)(x^q - 1)$; or, if we write h for $q + r$, and h_1 for q, and divide throughout by hh_1,

$$(x^h - 1)/h > (x^{h_1} - 1)/h_1. \tag{23}$$

On the other hand, if we divide (22) throughout by x^{q+r} and add $r - rx^{-r}$ to each side, we have $(q+r)(1 - x^{-r}) > r(1 - x^{-q-r})$; or, if we write h for $q + r$ and h_1 for r, and divide throughout by hh_1,

$$(1 - x^{-h_1})/h_1 > (1 - x^{-h})/h. \tag{24}$$

In each of these results h_1 is less than h. The inequality (23) is, as already stated, well known, and is recognized as highly important. (Cf. Chrystal, Algebra, II, 42–45.)

APPENDIX C.—The following inequality will be found more general and, so to speak, closer than the one (22) just discussed:

$$q\,(x^{q+r}-x^{-q-r}) \gtrless (q+r)(x^q-x^{-q}), \tag{25}$$

according as $x \gtrless 1$. In fact, it includes (22) as a special case. For, multiplying both sides of (25) by x^{q+r}, and writing $x^{\frac{1}{2}}$ for x, we have, according as $x \gtrless 1$,

$$q\,(x^{q+r}-1) \gtrless (q+r)(x^q-1)\,x^{\frac{1}{2}r}. \tag{26}$$

When $x > 1$, $x^{\frac{1}{2}r} > 1$, so that we have from (26)

$$q\,(x^{q+r}-1) > (q+r)(x^q-1),$$

which is merely (22) transposed. When $x < 1$, $x^{\frac{1}{2}r} > x^r$, so that we have from (26), changing signs,

$$q\,(1-x^{q+r}) > (q+r)(1-x^q)\,x^r,$$

whence, transposing,

$$rx^r\,(x^q-1) > q\,(x^r-1),$$

which is the same as (22), with q and r written each for the other. To demonstrate (25), we may remark that (m being an integer not greater than q), when $x \gtrless 1$, $x^{2q-m+1} \gtrless 1$, and therefore, multiplying both sides by x^m-1 (negative when $x<1$) and transposing, we have, for all values of x, the inequality $x^{2q+1}+1 > x^{2q-m+1}+x^m$. Summing q such inequalities, in which m has successively all integral values from 1 to q inclusive, we obtain

$$q\,(x^{2q+1}+1) > x^{2q}+x^{2q-1}+\ldots\ldots+x^2+x.$$

Multiplying both sides by $x-1$ (negative when $x<1$), we have now

$$q\,(x^{2q+2}+x-x^{2q+1}-1) \gtrless x^{2q+1}-x,$$

or by transposition, after dividing throughout by x^{q+1},

$$q\,(x^{q+1}-x^{-q-1}) \gtrless (q+1)(x^q-x^{-q}), \tag{27}$$

according as $x \gtrless 1$; so that (25) is true when $r=1$, for any positive integral value of q. Again, if (25) is true for any given integral value of r, it may be shown to be true for the value next greater, say $r+1$. For, if we write $(q+1)$ for q in (25) and multiply both sides by $q/(q+1)$, we have

$$\begin{aligned} q\,(x^{q+r+1}-x^{-q-r-1}) &\gtrless (q+r+1)\,q\,(x^{q+1}-x^{-q-1})/(q+1) \\ &\gtrless (q+r+1)(x^q-x^{-q}), \end{aligned}$$

since, by (27), $q(x^{q+1}-x^{-q-1})/(q+1) >< x^q - x^{-q}$. Hence, if (25) is true for any given value of r, it is true for the next higher value $r+1$; and since it is true for $r=1$, it is true for $r=2$, and so on for all other integral values. Proof for fractional exponents is to be supplied by writing $x^{1/p}$ for x. If, in (25), $\frac{1}{2}h$ be written for $q+r$, and $\frac{1}{2}h_1$, a smaller quantity than $\frac{1}{2}h$, for q, and both sides be divided by $\frac{1}{2}hh_1$, the general inequality takes this form:

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h >< (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1, \tag{28}$$

according as $x >< 1$. In this paragraph, as in the preceding, all the quantities concerned are real and positive, and h and h_1 are commensurable.

APPENDIX D.—If the series $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots$ be susceptible of interpretation, let us denote it by $\psi(z) = \phi_2(x)$, where $x = 1 + z$. It is to be proved that the sum of two such series, say $\phi_2(x) + \phi_2(u)$, is equal to $\phi_2(xu)$, provided the latter be also interpretable; or, if $w = u - 1$, that

$$\psi(z) + \psi(w) = \psi(zw + z + w). \tag{29}$$

The second member is a series of positive integral powers of the expression $z + w(1+z)$. Expanding these powers by the binomial theorem, assumed known for positive integral exponents, and separating the series forming the coefficient of $w^n(1+z)^n$, then expanding $(1+z)^n$ and multiplying the result by the coefficient just separated, and finally separating from the product the series forming the coefficient of $x^m w^n$, we find it to be, for all values of m and n greater than 0, the following expression multiplied by $(-1)^{n-1}/m!$:

$$(n-1)^{m-1)} - mn^{m-1)} + \ldots (-1)^r m^{r)}(n+r-1)^{m-1)}/r! \ldots (-1)^m(n+m-1)^{m-1)},$$

where $x^{k)}$ represents $x(x-1)(x-2)\ldots(x-k+1)$. This expression is equal to 0, by a well-known theorem in finite differences, of which I have given an algebraic proof in a preceding paper ("Algebraic Proof of a Certain Series," *ante*, p. —). Therefore all terms in $x^m w^n$, that is to say, all terms which contain both x and w, vanish. The terms remaining, which contain z alone and w alone, are respectively $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots = \psi(z)$, and $w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \ldots = \psi(w)$, so that (29) is proved true. The proof thus given is valid for real quantities when the three quantities concerned, namely, z, w and $zw + z + w$ are such that no one of them is greater than 1 and all are greater than -1. In such cases

x, u and xu are all greater than 0 and not greater than 2. Within these limits, therefore,

$$\phi_2(x) + \phi_2(u) = \phi_2(xu)\,, \tag{30}$$

by (29), since $xu - 1 = (1 + z)(1 + w) - 1 = zw + z + w$. If, for any value of x greater than 2, we define $\phi_2(x)$ to be the series denoted by $-\phi_2(1/x)$, the property shown in (30) will still hold true. For, when x and u are both greater than 1, $\phi_2(1/x) + \phi_2(1/u) = \phi_2(1/xu)$, and on changing signs we have (30). If $x > 1$, $u < 1$, $xu < 1$, we have $\phi_2(xu) + \phi_2(1/x) = \phi_2(u)$, or $\phi_2(u) + \phi_2(x) = \phi_2(xu)$ by transposition. Or, if $x > 1$, $u < 1$, $xu > 1$, $\phi_2(1/xu) + \phi_2(u) = \phi_2(1/x)$, which gives the same result. A portion of this paragraph has been anticipated in my earlier paper, "An Essay on the Calculus of Enlargement," *American Journal of Mathematics*, II, 122, 123.